BRANCHED SURFACES AND COLORED PATTERNS

JUAN ESCUDERO

A cell complex is defined in the analysis of the topological invariants of tiling spaces. In some cases the complex contains collared tiles. The representation of the corresponding branched surface can be done by assigning colors to the collared tiles. This allows to distinguish tiles with the same shape but different edge identifications.

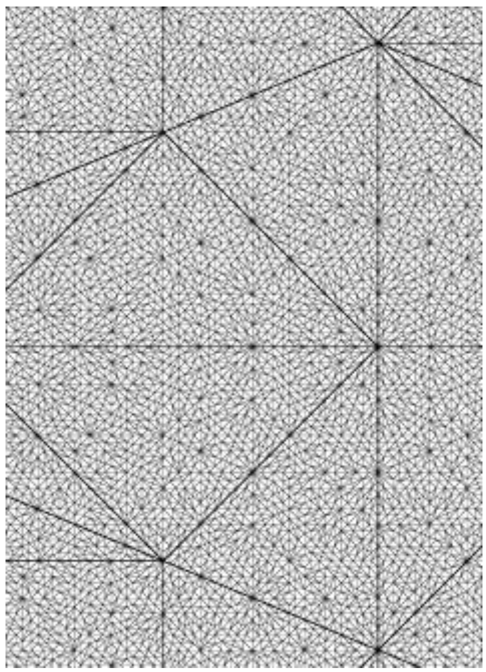

Fig.1 Level-4 supertiles for the octagonal tiling

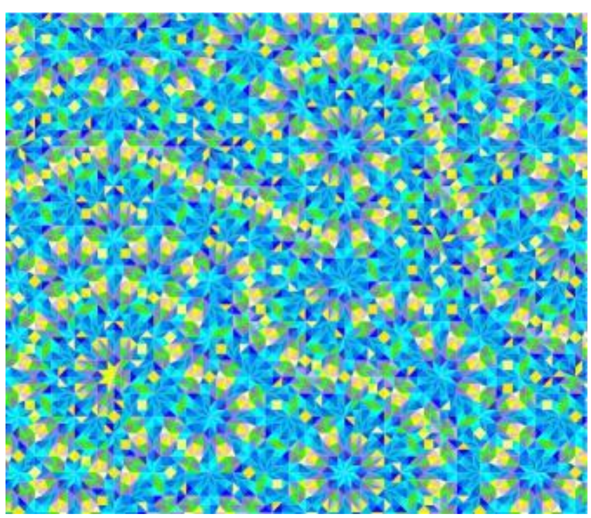

Fig 2. A fragment of "Branched Manifold"

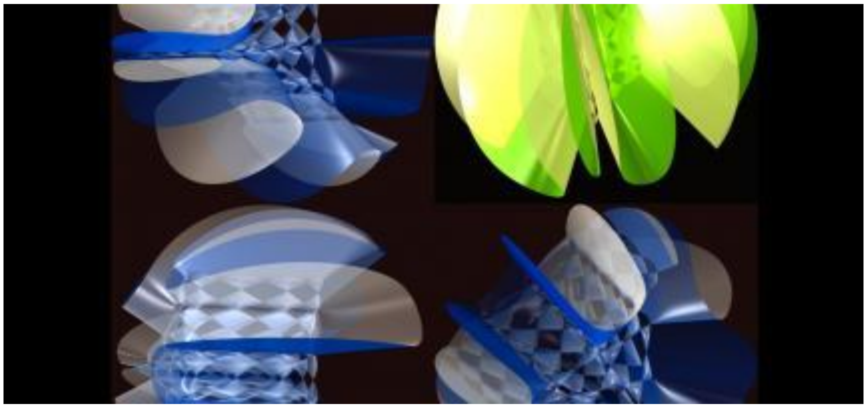

Fig 3. "Nueve y 220-A" is based on a nodal surface with degree nine and cyclic symmetry.

1.-Introduction

Artists, scientists and mathematicians share instinctive feelings about order and disorder. One of the fields where this is apparent is the mathematical theory of long range aperiodic order, because of its implications in the arts.

Aperiodic tilings are geometric objects lying somewhere between periodicity and randomness. In the 1960´s Wang and Berger introduced aperiodic sets of tiles in the treatment of certain problems in logic. The question was whether or not it is possible to determine algorithmically if given a set of tiles they tile the plane. The cardinal of the tile sets was very high and examples with few prototiles were constructed later by Robinson, Penrose, Ammann, and others. Since the discovery of quasicrystals in the 1980´s, the generation of ideal quasiperiodic structures has been a problem studied mainly by mathematicians and physicists.

Recently it has been suggested that aperiodic order already was present in the medieval islamic architecture. [8] Periodic and non-periodic girih (geometric star-and-polygon, or strapwork) patterns were on the basis of the designs. In particular by using certain substitutions in five girih tiles, a pattern on the Darb-i Imam shrine (Isfahan, Iran, 1453 C.E.) can be mapped into a decagonal quasicrystalline Penrose pattern with few defects. The girih tiles have the shape of the decagon, pentagon, hexagon, bowtie and rhombus. They can be seen in one of the panels of the Topkapi scroll (Topkapi Palace Museum in Istanbul), drafted by Islamic designers to transmit architectural procedures. The authors in [8] also claim that a selfsimilarity transformation, or subdivision of large girih tiles into smaller ones, was known by islamic architects.

In the 20th century, there are also many examples of non-periodic order in the arts. Xenakis, while working as an engineer in Le Corbusier's office, was responsible for the design of the undulating glass panels at the facade of the monastery of St Marie de La Tourette. Four one-dimensional tiles in golden proportion and their combinatorial distributions were the constructive units. [10] At the same epoch he employed Fibonacci series to organize the temporal sections in Metastasis, a work based also on ruled surfaces in the form of continuous massive glissando structures. The idea of transforming graphics into sound was elaborated by Xenakis at the UPIC system in the late 1970´s.

Aperiodic order is present also in the design of more recent architecture. Penrose tilings, with tiles appearing in ten different orientations, are used in the Royal Institute of Technology RMIT and the pinwheel tiling, with tiles appearing in all rotation angles, in the Federation Square buildings, both in Melbourne, Australia. Obviously in many cases, the use of patterns with interesting mathematical properties does not necessarily give results aesthetically appealing.

From a mathematical point of view, the appropriate space in pattern analysis is not the original surface but a folded version of it which is called the orbifold. [1] The set of points of the same kind is called the orbit of the symmetry group and the folding takes all the points of the same kind to a single point. Repeating patterns can be folded into an orbifold on some surface. The description of manifolds in two dimensions is often done by identifying some edges of simpler surfaces. Deterministic and random aperiodic tilings in two and three-dimensional manifolds have been presented in the past few years (see [5] and references therein).

There are several methods for constructing aperiodic tilings: cut-and-project methods, substitutions and matching rules. Substitution tilings grow by iteration of a set of inflation rules applied to a given set of prototiles. A tiling space can be seen as the set of tilings that locally look like translates of a fixed tiling. For the analysis of the cohomology of tiling spaces a type of cell complex is defined. [9] For each particular case the complex contains a copy of every kind of tile that is allowed, with some edges identified, and the result is a branched surface that can not be represented properly in three dimensions. A way to get an idea about it is to generate a pattern where the basic polygons with the same shape, color and orientation represent the same tile in the complex. [7] The topological interpretation is that if somewhere in the pattern a tile shares an edge with another tile, then those two edges are identified. The goal in the geometric representation is to visualize in some way the space unfolded without need of supplementary dimensions.

In the visual and sound arts, this type of constructions have potential interest as a system of reference in constrictive preforming for channeling the expressive energies.

2.-A branched surface associated to an octagonal tiling space.

In a substitution tiling the pattern obtained after applying n times the inflation rules to a given prototile is called a level-n supertile. A substitution is said to force the border if there is a positive integer n such that any two level-n supertiles of the same type have the same pattern of neighboring tiles. Tiles labeled by the pattern of their neighbors are called collared tiles. When the substitution does not force the border, collared tiles can be used for the study of a type of topological invariants known as Cech cohomology groups. [9,7] In what follows I discuss how the procedure is applied to one of the octagonal tilings introduced in Escudero. [3] The study of its cohomology motivates the generation of colored aperiodic tessellations which represent branched surfaces. [6] In contrast to other well-known octagonal examples with the silver mean as scaling factor, like the Ammann-Benker patterns, this substitution does not force the border.

A vertex configuration is a set of tiles sharing a vertex. The first step in the construction consists in analyzing the dynamics of the vertex configurations. The tiling has the property of finite local complexity, which means that for some positive real number R, the tiling contains, up to congruence, only finitely many local patches of diameter less than R. Also it is uniquelly ergodic, namely, it has well defined patch frequencies. In general after n inflation steps all the vertex configurations are transformed into a finite subset. In this case it is formed by just two vertex configurations. This is the set of vertices taken to form the cell complex.

The tilings with eight-fold symmetry in [3] have four triangular prototiles represented by the letters A,B,C,D. Their edge sets are: A(a,b,b), B(g,d,e), D(g,b,b), E(b,b,z), wherea,b,g,d,e,z represent the edges having lengths 2c1,1, 2c2, 2c1c2, 2c2c3, 2c3, respectively, with ck =cos(k pi / 8).

Iteration of the inflation rules applied to a given prototile shows that there are, up to mirror reflection, forty-one vertex configurations. After four inflation steps all of them are transformed into just two, that we label 1 and 2. This can be seen in Fig.1, where the superposition of two patterns separated by four inflation steps is shown. The vertices transform into themselves under the application of the inflation rules: 1->1, 2->2. In Figs.1 and 2 we can see both: the vertex 1 has star shape with sixteen A-type prototiles (blue), and the vertex 2 has ten A-type (eight of them yellow-green) and two of type D (green).

The edges and tiles appear in eight different orientations. By analyzing level-4 supertiles, we get the sets of possible collared tiles and edges. Having in mind the different edges and vertices we can construct a cell complex which contains 25 collared prototiles, up to mirror reflection and orientation. In order to distinguish the prototiles with the same shape we assign them colors. More than one hundred colored vertices from the forty-one initial vertex configurations appear. A fragment of the pattern representing the corresponding branched surface can be seen in Fig.2.

The collared pattern can be described in terms of formal language theory, more precisely, with the help of Lindenmayer systems. [3] The words characterize the tilings in a unique way. The word production rules are defined with the intention to describe finite patterns by word sequences as a kind of symbolic dynamical system, which is very "natural" in one dimensional substitutions like the Fibonacci sequence. In one dimension if two letters appear consecutively then the corresponding tiles appear together in the geometric representation. In order to generalize this to two and higher dimensions one has to introduce a bracket structure and the letters in the alphabet represent oriented prototiles. The allowed words in the formal language then are of the type ((ABC)(FA)(DE)). The geometric interpretation is as follows: A and B appear together in the word and there is only one way to "glue" the corresponding oriented tiles edge-to-edge. The same applies to the supertiles represented by (ABC) and (FA) or to (((word1))) with (((word2))) ... if they appear consecutively in a given word (notice that C and F will not appear, in general, adjacent in the geometric structure). A bracket belonging to the alphabet has to be interpreted not as a tile, but as a way to group tiles to get supertiles. The use of brackets mimics the hierarchical structure and it seems that they are not avoidable (without losing the result that if two letters representing tiles appear consecutively then the corresponding tiles can be glued in only one way). The model has the advantage that can be applied to higher dimensions as well.

3. Concluding remarks

In the construction of the cell complex another point of interest is to analyze the ways color can interact with the symmetries of the pattern. For a discussion of the mathematics of color symmetry see. [1] Here there is a freedom in the color selection and the final results can be very different, due to the emphasis in distinct geometric substructures. When we take into account the set of the patterns obtained by all the possible color selections we are approaching to a metaphor of the concept of rhizome which is made of plateaus. Each pattern would be an image of a plateau or “continuous, self-vibrating region of intensities whose development avoids any orientation toward a culmination point or external end.” [2] Also each pattern is related to any other pattern and can be generated starting in a limitless number of ways. There is a principle of connection and heterogeneity in the sense that any local geometric configuration appears in some other place, in fact, in infinite places when the plane filling structure is considered. The pattern is made of lines and occupy all the dimension of a “plane of consistency” following a principle of multiplicity. Each plateau here would have a strong principal unity of root-tree type because its generation is the result of successive iteration of a large set of inflation or substitution rules in a Lindenmayer system. However the whole set of colored patterns does not have this arborescent characteristic.

The basic symmetries are continuously broken and have to be perceived in a dynamical way as would be the case if temporal phenomena were embedded. There are also various levels of perception depending on the distance of observation. The image can then be seen as a kind of nomad place. While we contemplate it we travel through a space in constant change, where local configurations of just four shapes, like a ritornello, always reappear but in different surroundings. This property is preserved when we extend the pattern to infinity, but we can have a sensation of it by observing a finite fragment.

The simplicial arrangements of lines given in [5] are the basis for the derivation of non-periodic planar tilings with any symmetry. The analysis of their associated topological invariants provides a rich source of branched surfaces generation. In addition, certain families of simple subarrangements can be used for the construction of algebraic surfaces with many nodal singularities that can be represented in 3D. In Fig.3 it is shown a work based on a surface corresponding to a polynomial of degree nine obtained from one of the arrangements with 18 lines in [5] and containing the seven prototiles of a series of tilings introduced by the author in 1998. It has cyclic symmetry and 220 real nodes.

In the time domain, substitution tilings and their appearance in the field of astronomy have been on the basis of the formal procedures in several instrumental, vocal and computer generated works, where time harmonizations and sound synthesis derived from spectra of aperiodic ordered sequences play a central role. [4] One of the pieces where this techniques are present is *Yod*, for 6 percussionists and computer, performed by the austrian group Studio Percussion Graz at the 2005 ISCM World Music Days in Zagreb. On the other hand certain identifications leading to quotient spaces and orbifolds have been commonplace in musical practice. A recent work, where this and other concepts of combinatorial topology are explored as part of the precompositional materials is *Los límites móviles del agua* for two pianos, which the Ensemble Surplus plans to perform in Freiburg, Germany.

Both the visual and sound works must be regarded as projections from the same rhizomatic space. They are just manifestations of some of its infinite plateaus.

References and Notes:

1. *J.H. Conway, H. Burgiel, C. Goodmann-Strauss, The Symmetries of Things (Wellesley: A.K.Peters, 2008).*
2. *G. Deleuze and F. Guattari, A Thousand Plateaus: Capitalism and Schizophrenia (Minnesota: University of Minnesota Press,1987).*
3. *J.G. Escudero, "A Construction of Inflation Rules for Pisot Octagonal Tilings," in Int. J. Mod. Phys. B 17 (2003): 2925-31.*
4. *J.G. Escudero, "Harmonizations of Time with Non Periodic Ordered Structures in Discrete Geometry and Astronomy," in Computer Music Modeling and Retrieval, LNCS 3310 (Berlin: Springer Verlag, 2005), 107-18.*
5. *J.G. Escudero, "Random tilings of spherical 3-manifolds," J. Geom. Phys. 58 (2008): 1451-64.*
6. *J.G. Escudero, "Branched Manifold," digital print, 2002-2010.*
7. *J.G. Escudero, "Randomness and topological invariants in pentagonal tiling spaces," Discrete Dyn. Nat. Soc, Article ID 946913 (2011): 1- 23.*
8. *P.J. Lu and P.J. Steinhardt, "Decagonal and quasicrystalline tilings in medieval Islamic architecture," in Science 315 (2007): 1106-10.*
9. *L. Sadun, Topology of Tiling Spaces (Providence, RI: AMS, 2008).*
10. *I. Xenakis, Musique de l´architecture (Paris: Ed. Parenthèses, 2005).*